

Of ants and urns: estimation of the parameters of a reinforced random walk and application to ants behavior

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Abstract

In applications, reinforced random walks are often used to model the influence of past choices among a finite number of possibilities on the next choice. For instance, in an urn model with balls of two colors (red and green), the probability of drawing a red ball at time $n + 1$ is a function of the proportion of red balls at time n , and this proportion changes after each draw. The most famous example is Polya's urn. This function, called the choice function, can be known up to a finite dimensional parameter. In this paper, we study two estimators of this parameter, the maximum likelihood estimator and a weighted least squares estimator which is less efficient but is closer to the calibration techniques used in the applied literature. In general, the model is an inhomogeneous Markov chain and because of this inhomogeneity, it is not possible to estimate this parameter on a single path, even if it were infinite. Therefore we assume that we can observe i.i.d. experiments, each of a predetermined finite length. This is coherent with the experimental set-up we are interested in: the selection of a path by laboratory ants. We study our estimators in a general framework and then restrict to a particular model in order to do a simulation study and an application to a an experiment with ants. Our findings do not contradict the biological literature, but we give statistical significance to the values of the parameter found therein. In particular we compute Bootstrap confidence intervals.

1 Introduction

A colony of ants is able to create an optimal dynamic network of paths to and from several food sources. This network is made of a chemical substance called the trail pheromone laid by the ants and to which they are attracted. While progressing along this network, they are faced with a succession of bifurcations. At each of these bifurcations, the decision to choose a branch is mostly based on the concentration of the trail pheromone. The

collective exploratory scheme is thus determined by the individual and local behavior of trail pheromone laying and following.

The interest of understanding this mechanism is obvious, but extremely difficult since it is nearly impossible to detect and measure the quantity of trail pheromone laid by each ant. It is generally assumed that each ant lays the same average amount of pheromone. The Argentine ant (*Linepithema humile*) is known to have approximately this behavior. Therefore, in experiments, Argentine ants are often used, and the number of passages through the branches is used to obtain information on the quantity of pheromone laid and the attractiveness of the pheromone (see [CDF⁺01] for a review).

The choice of each ant is often modeled as a random function of the previous choices. After N passages, the probability $prob_N^R$ that an ant facing a bifurcation with two branches chooses the right branch is given by a function $f : \mathbb{N} \times \mathbb{N} \rightarrow [0, 1]$ of the numbers N_R and N_L of passages through the right and left branches, respectively:

$$prob_N^R = f(N_R, N_L) \quad (1.1)$$

One common choice for this function (first introduced by [DAGP90]) is

$$prob_N^R = \frac{(c + N_R)^\alpha}{(c + N_R)^\alpha + (c + N_L)^\alpha} . \quad (1.2)$$

The parameter c is called the intrinsic attraction of each branch. Rewriting (1.2) as

$$prob_N^R = \frac{(1 + N_R/c)^\alpha}{(1 + N_R/c)^\alpha + (1 + N_L/c)^\alpha} .$$

we see that the parameter c may be considered as the inverse of the quantity of pheromone laid by each ant. That is, the smaller c is, the more pheromone is laid and the more attractive is the branch. The parameter α induces a non linearity and amplifies or diminishes the attractiveness of the pheromone. Whereas c has a concrete meaning and might eventually be measured physically, α is a purely "behavioral" parameter which cannot be directly measured. Throughout the paper, α and c will be referred to as the intensification and attractiveness parameters, respectively. It must be kept in mind that c is actually the inverse of the attractiveness, that is a large value of c corresponds to a low attractiveness of the branch.

The main feature of this model is that if $\alpha > 1$ then selection of a branch eventually occurs with probability 1. After a random but finite number of passages, a branch will be selected and the other one will be left. Theoretically, this phenomenon occurs whatever the value of the parameter c . However, it is clear that the value of c must have an influence on the ants behavior: a very large value of c makes the selection of a branch much more difficult, and a small value of c makes the first branch chosen much more attractive than the other. Biological experiments have been made to evaluate these two parameters. The method of used to evaluate them is usually a method of calibration by simulation. Values of α

greater than one have been proposed, but confidence interval are usually not provided so the statistical significance of the numbers obtained is uncertain.

In probability theory, the model described by equation (1.1) is called a reinforced random walk and it is the goal of this paper to propose and investigate estimators of the choice function (1.2). In view of possible applications outside the field of quantitative ethology, we will consider a general parametric model: we will assume that the choice function f depends on an unknown parameter θ to be estimated; that is

$$prob_N^R = f(\theta, N_R, N_L) \tag{1.3}$$

The reinforced random walk model (1.2) corresponds to $\theta = (\alpha, c)$. This model is used for modeling neuron polarity (see [KK01]). Other choice functions are used in other fields such as economics, e.g. to study the creation of a technology monopoly [DFM02], in news recommendation systems [PP12]...

In Section 2, we present the statistical set-up that we will use. We assume that we observe N independent experiments, each consisting of a path of fixed length n of a reinforced random walk. This framework is necessary since in most models, it is not possible to obtain consistent estimators of the parameters with only one path, even if its length n increases to infinity. This will be rigorously explained in Section 3.3. This model is the subject of a vast literature in probability theory, but also in many other applied fields. For instance

In Section 2.1 we focus on the maximum likelihood estimator (MLE) in the model (1.3). One important result is that under the usual regularity assumptions on the model, the MLE is consistent, asymptotically normal and efficient in the sense of Fisher.

The MLE is sometimes numerically hard to compute, and the likelihood of some general models might even be impossible to write explicitly. In Section 2.2, we introduce a weighted least squares estimator WLSE. Similarly to the MLE, we prove that under regularity conditions, these estimators are consistent and asymptotically normal, but generally not efficient. These least squares estimators are closer to the calibration methods used in the biological literature, where least squares are used to fit certain curves.

In Section 3, we apply the previous results to the classical Reinforced Random Walk model (1.2). We show in Section 3.3 that the Fisher information of one single path remains bounded when the length n of the path tends to infinity, for most values of the parameters. This proves that no consistent estimation is possible based on a single path.

In Section 3.4, we study numerically the maximum likelihood and least squares estimators of the parameters of the reinforced random walk model. Finally, these estimators are applied to a real (ants) life experiment. Results are reported in Section 3.5. Some concluding remarks are in Section 4. The mathematical proofs are postponed to Section 5

2 Estimation of the parameters of a choice function

Let us first write precisely the model we are going to investigate. We assume $\{X_k, k \geq 1\}$ is a sequence of Bernoulli random variables (representing for instance the color of a ball drawn in an urn) and that there exists a function $f_0 : \mathbb{N}^2 \rightarrow [0, 1]$ such that for all integers $0 \leq i \leq k$,

$$\mathbb{P}(X_{k+1} = 1 \mid \mathcal{F}_k) = f_0(Z_k, k - Z_k) .$$

where $Z_0 = 0$ and for $k \geq 1$, $Z_k = X_1 + \dots + X_k$ and \mathcal{F}_k is the sigma-field generated by Z_0, X_1, \dots, X_k . The random walk $\{Z_k, k \geq 0\}$ is an inhomogeneous Markov chain. For $n \geq 1$ and a sequence $(e_1, \dots, e_n) \in \{0, 1\}^n$, applying the Markov property, we obtain

$$\begin{aligned} \mathbb{P}(X_1 = e_1, \dots, X_n = e_n) &= \prod_{k=0}^{n-1} f_0(e_1 + \dots + e_k, k - e_1 - \dots - e_k)^{e_{k+1}} \\ &\quad \times \{1 - f_0(e_1 + \dots + e_k, k - e_1 - \dots - e_k)\}^{1-e_{k+1}} . \end{aligned} \quad (2.1)$$

In order to compute the distribution of Z_k , we introduce some notation. Let \mathcal{S}_k be the set of sequence of length $k+1$ of integers i_0, \dots, i_k such that $i_0 = 0$ and $i_j - i_{j-1} \in \{0, 1\}$ for $j = 1, \dots, k$. For $i \leq k$ let $\mathcal{S}_k(i) = \{(i_0, \dots, i_k) \in \mathcal{S}_k \mid i_k = i\}$

$$\mathbb{P}(Z_k = i) = \sum_{(i_0, \dots, i_k) \in \mathcal{S}_k(i)} \prod_{q=0}^{k-1} f_0(i_q, q - i_q)^{i_{q+1} - i_q} (1 - f_0(i_q, q - i_q))^{1 - i_{q+1} + i_q} . \quad (2.2)$$

2.1 Maximum likelihood estimator (MLE)

As mentioned previously, the model studied is an inhomogeneous Markov chain. Because of this inhomogeneity, it is possible that the parameters cannot be consistently estimated on one single path. We will prove this rigorously for the reinforced random walk model in Section 3.3. Therefore we assume that we observe N experiments, each consisting of a path of length n of the model (2.1). For $j = 1, \dots, N$ and $k = 1, \dots, n$, let $X_k^j \in \{0, 1\}$ denote the color of the k -th ball drawn in the j -th experiment, $Z_0^j = 0$ and $Z_k^j = \sum_{\ell=1}^k X_\ell^j$, $k \geq 1$ be the total number red balls drawn at time k during the j -th experiment, so that $X_k^j = Z_k^j - Z_{k-1}^j$. In all the paper, n will be fixed and our asymptotic results will be obtained with N (the number of experiments) tending to ∞ .

Let Θ be a compact subset \mathbb{R}^d with non empty interior and $f : \Theta \times \mathbb{N}^2 \rightarrow (0, 1)$ be a continuous function. We assume that there exists $\theta_0 \in \Theta$ such that $f_0(\cdot, \cdot) = f(\theta_0, \cdot, \cdot)$, i.e. for $k = 0, \dots, n-1$ and $i = 0, \dots, k$,

$$\mathbb{P}(X_{k+1}^j = 1 \mid Z_k^j = i) = f(\theta_0, i, k - i) .$$

Since the N experiments are independent, the Markov property of each path yields the following multiplicative form for the likelihood $V_N(\theta)$:

$$V_N(\theta) = \prod_{j=1}^N \prod_{k=0}^{n-1} f(\theta, Z_k^j, k - Z_k^j)^{X_{k+1}^j} \{1 - f(\theta, Z_k^j, k - Z_k^j)\}^{1-X_{k+1}^j} .$$

The log-likelihood function L_N based on N path, is thus given by

$$L_N(\theta) = \sum_{j=1}^N \sum_{k=0}^{n-1} \{X_{k+1}^j \log f(\theta, Z_k^j, k - Z_k^j) + (1 - X_{k+1}^j) \log \{1 - f(\theta, Z_k^j, k - Z_k^j)\}\} .$$

Let $\hat{\theta}_N$ be the maximum likelihood estimator of θ_0 , that is,

$$\hat{\theta}_N = \arg \max_{\theta \in \Theta} L_N(\theta) . \quad (2.3)$$

Define $L(\theta) = N^{-1} \mathbb{E}[L_N(\theta)]$. Then, by the Markov property, we have

$$\begin{aligned} L(\theta) &= \sum_{k=0}^{n-1} \mathbb{E} [f_0(Z_k, k - Z_k) \log f(\theta, Z_k, k - Z_k) \\ &\quad + \{1 - f_0(Z_k, k - Z_k)\} \log \{1 - f(\theta, Z_k, k - Z_k)\}] . \end{aligned} \quad (2.4)$$

For any function g defined on Θ , we denote \dot{g} and \ddot{g} the gradient and Hessian matrix with respect to θ , $\partial_s g$ its partial derivative with respect to the s -th component θ_s of θ , $1 \leq s \leq d$ and A' the transpose of the vector or matrix A .

Assumption 2.1.

- (i) (Regularity) The set Θ is a compact subset of \mathbb{R}^d . For $0 \leq i \leq k \leq n-1$, $f_0(i, k-i) > 0$ and the function $\theta \rightarrow f(\theta, i, k-i)$ is twice continuously differentiable on Θ .
- (ii) (Identifiability) If $f(\theta_1, i, k-i) = f(\theta_2, i, k-i)$ for all $0 \leq i \leq k \leq n-1$, then $\theta_1 = \theta_2$.
- (iii) The $n(n-1)$ -dimensional vectors $\{\partial_s f(\theta_0, i, k-i), 0 \leq i \leq k \leq n-1\}$, $1 \leq s \leq d$, are linearly independent in $\mathbb{R}^{n(n-1)}$.

Assumption 2.1 ensures that θ_0 is the unique maximizer of L and that the Fisher information matrix

$$\mathcal{I}_n(\theta_0) = -\ddot{L}(\theta_0) = \sum_{k=0}^{n-1} \sum_{i=0}^k \frac{\mathbb{P}(Z_k = i)}{f_0(i, k-i) \bar{f}_0(i, k-i)} \dot{f}_0(i, k-i) \dot{f}_0(i, k-i)'$$

is invertible, where we denote $\dot{f}_0(i, k-i) = \dot{f}(\theta_0, i, k-i)$ and $\bar{f}_0(i, k-i) = 1 - f_0(i, k-i)$. Let $\mathcal{N}(m, \Sigma)$ denote the Gaussian distribution with mean m and covariance Σ .

Theorem 2.2. *If Assumptions 2.1-(i)-(ii) hold then the estimator $\hat{\theta}_N$ is a consistent estimator of θ_0 . If moreover Assumption 2.1-(iii) holds and θ_0 is an interior point of Θ , then as N tends to ∞ ,*

$$\sqrt{N}(\hat{\theta}_N - \theta_0) \xrightarrow{w} \mathcal{N}(0, \mathcal{I}_n^{-1}(\theta_0)) .$$

The proof is at the end of Section 5.2. It is the consequence of a more general result stated and proved therein.

2.2 Weighted least squares estimator (WLSE)

In some models, the maximum likelihood estimator may be numerically unstable and long to compute. An alternative is to use a weighted least squares estimator to fit the parameter θ to the observations. For $0 \leq i \leq k \leq n-1$, define

$$a_N(i, k-i) = \frac{1}{N} \sum_{j=1}^N \mathbb{1}_{\{Z_k^j=i\}} , \quad p_N(i, k-i) = \frac{\frac{1}{N} \sum_{j=1}^N \mathbb{1}_{\{Z_k^j=i\}} X_{k+1}^j}{a_N(i, k-i)} , \quad (2.5)$$

with the convention $\frac{0}{0} = 0$. The quantity $a_N(i, k-i)$ is the empirical probability that i red balls have been drawn at time k and $p_N(i, k-i)$ is the empirical conditional probability that a red ball is again chosen at time $k+1$ given i red balls were drawn at time k .

We further define $q_N(i, k-i) = 1 - p_N(i, k-i)$ and $\bar{f}(\theta, i, k-i) = 1 - f(\theta, i, k-i)$. Then, assuming that $0 < p_N(i, k-i) < 1$ for all $i \leq k$, we have

$$\begin{aligned} \frac{1}{N} L_N(\theta) &= \frac{1}{N} \sum_{j=1}^N \sum_{k=0}^{n-1} \{X_{k+1}^j \log f(\theta, Z_k^j, k-Z_k^j) + (1-X_{k+1}^j) \log \bar{f}(\theta, Z_k^j, k-Z_k^j)\} \\ &= \sum_{k=0}^{n-1} \sum_{i=0}^k a_N(i, k-i) \{p_N(i, k-i) \log f(\theta, i, k-i) + q_N(i, k-i) \log \bar{f}(\theta, i, k-i)\} \\ &= \sum_{k=0}^{n-1} \sum_{i=0}^k a_N(i, k-i) \{p_N(i, k-i) \log p_N(i, k-i) + q_N(i, k-i) \log q_N(i, k-i)\} \\ &\quad + \sum_{k=0}^{n-1} \sum_{i=0}^k a_N(i, k-i) p_N(i, k-i) \log \left(1 + \frac{f(\theta, i, k-i) - p_N(i, k-i)}{p_N(i, k-i)} \right) \\ &\quad + \sum_{k=0}^{n-1} \sum_{i=0}^k a_N(i, k-i) q_N(i, k-i) \log \left(1 - \frac{f(\theta, i, k-i) - p_N(i, k-i)}{q_N(i, k-i)} \right) . \end{aligned}$$

Since for each $0 \leq i \leq k \leq n-1$, the probability $p_N(i, k-i)$ converges almost surely to $f_0(i, k-i)$ and $a_N(i, k-i)$ converges to $\mathbb{P}(Z_k = i)$ as N tends to infinity, a second order

Taylor expansion yields

$$\begin{aligned} \frac{1}{N} L_N(\theta) \sim -S_N(\theta) + \sum_{k=0}^{n-1} \sum_{i=0}^k a_N(i, k-i) \{ p_N(i, k-i) \log p_N(i, k-i) \\ + q_N(i, k-i) \log q_N(i, k-i) \} , \end{aligned}$$

in probability, with

$$S_N(\theta) = \frac{1}{2} \sum_{k=0}^{n-1} \sum_{i=0}^k a_N(i, k-i) \frac{(p_N(i, k-i) - f(\theta, i, k-i))^2}{p_N(i, k-i) q_N(i, k-i)} . \quad (2.6)$$

Thus we may expect that maximizing the likelihood is equivalent to minimizing $S_N(\theta)$. Define

$$\hat{\theta}_N^S = \arg \min_{\theta \in \Theta} S_N(\theta) .$$

The weighted least squares estimator is the value of the parameter θ which provides the best (in the sense of the least squares) fit of the curve $(i, k) \rightarrow f(\theta, i, k-i)$ to the points $((i, k), \hat{p}_N(i, k-i))$, for $0 \leq i \leq k \leq n-1$. If the parametrization is trivial (tautological), that is if $\theta = \{f(i, k-i), 0 \leq i \leq k \leq n-1\}$, then $\hat{\theta}_N^S = p_N$ is the maximum likelihood of f . See Remark 5.2 below. Moreover, least squares estimators are often used by practitioners because they are intuitive and easily implemented (and usually faster than the MLE).

We will prove that the estimator $\hat{\theta}_N^S$ is consistent and asymptotically normal and efficient in the sense that it has the same asymptotic variance as the maximum likelihood estimator. However, we will also see in Section 3.4 that in the case of the strongly reinforced random walk model, it is also numerically unstable, because the empirical probabilities $p_N(i, k-i)$ and $q_N(i, k-i)$ can be very small. Therefore, at the cost of a loss of asymptotic efficiency, we will use in practice a modified WLS estimator which has better numerical performances. To establish a general theory, we consider an arbitrary sequence $w_N(i, k-i)$ of weights and define the contrast function

$$W_N(\theta) = \sum_{k=0}^{n-1} \sum_{i=0}^k w_N(i, k-i) \{ p_N(i, k-i) - f(\theta, i, k-i) \}^2 , \quad (2.7)$$

and the estimator

$$\hat{\theta}_N^W = \arg \min_{\theta \in \Theta} W_N(\theta) . \quad (2.8)$$

Theorem 2.3. *Let Assumption 2.1 hold and assume that the weights w_N converge almost surely to a sequence of positive weights w_0 . Then $\hat{\theta}_N^W$ is a consistent estimator of θ_0 and $\sqrt{N}(\hat{\theta}_N^W - \theta_0)$ converges weakly (as N tends to ∞) to a Gaussian distribution with zero mean and definite positive covariance matrix $\Sigma_n(\theta_0)$.*

The proof is in Section 5.2. We give an explicit expression for $\Sigma_n(\theta_0)$ in two particular cases

- If $w_N(i, k-i) = p_N^{-1}(i, k-i)q_N^{-1}(i, k-i)a_N(i, k-i)$, then $\Sigma_n(\theta_0) = \mathcal{I}_n(\theta_0)^{-1}$.
- In Section 3.4, we will choose $w_N(i, k-i) = a_N(i, k-i)$. Then $\Sigma_n(\theta_0) = \ddot{W}^{-1}(\theta_0)H(\theta_0)\ddot{W}^{-1}(\theta_0)$, where $\ddot{W}(\theta_0)$ is the Hessian matrix of the limit $W(\theta_0)$ of W_N and $H(\theta_0)$ is a positive definite matrix defined in (5.3). The limiting covariance matrix $\Sigma(\theta_0)$ is no longer equal to the inverse of the Fisher information, i.e. this version of the WLSE is not asymptotically efficient.

3 Reinforced random walk model

In this section, we consider a model with a binary choice at each step, e.g. an urn with red and green balls. We specify the function f which gives the conditional probability of choosing a red ball given the previous draws. For $\theta = (\alpha, c) \in (0, \infty)^2$ and all integers $0 \leq i \leq k$, we define

$$f(\theta, i, k-i) = \frac{(c+i)^\alpha}{(c+i)^\alpha + (c+k-i)^\alpha} . \quad (3.1)$$

This yields the following model of reinforced random walk:

$$\mathbb{P}(X_{k+1} = 1 | \mathcal{F}_k) = f(\theta, Z_k, k - Z_k) = \frac{(c + Z_k)^\alpha}{(c + Z_k)^\alpha + (c + k - Z_k)^\alpha} .$$

For $\alpha = 1$, this is the Polya urn model with c red balls and c black balls at time 0. This process has been exhaustively investigated in the literature. We recall here its main features.

Theorem 3.1. (i) If $\alpha < 1$, then

$$\lim_{n \rightarrow \infty} \frac{Z_n}{n} = \frac{1}{2} \text{ a.s.}$$

(ii) If $\alpha = 1$, then $n^{-1}Z_n$ converges almost surely to a random limit with a Beta(c, c) distribution with density $x \rightarrow \Gamma^{-2}(c)\Gamma(2c)x^{c-1}(1-x)^{c-1}$ with respect to Lebesgue's measure on $[0, 1]$, and Γ is the Gamma function.

(iii) If $\alpha > 1$, then eventually only one color will be drawn, i.e.

$$\exists i \in \{0, 1\} , \exists n_0 \in \mathbb{N} , \forall n > n_0 , X_n = i .$$

Moreover, there exist constants $Q^* \in (0, \infty)$ and $0 < C_1 \leq C_2 < \infty$ such that

$$\lim_{l \rightarrow \infty} l^\alpha \mathbb{P}(Q_\infty = l) = Q^* , \quad (3.2)$$

$$C_1 x^{-(\alpha+1/\alpha-2)} \leq \mathbb{P}(T_\infty > x) \leq C_2 x^{-(\alpha+1/\alpha-2)} , \quad (3.3)$$

where Q_∞ is the total number of draws of balls of the color which is eventually not selected and T_∞ is the last time such a ball is drawn.

The case $\alpha < 1$ is due to [Tar11]; the case $\alpha > 1$ to [Lim03] and the convergences (3.2,3.3) to [CL09]. See [Pem07] for the case $\alpha = 1$.

If $\alpha \neq 1$ the asymptotic behavior of the proportion of red balls depends only on α and not on c . If $\alpha > 1$, one color is eventually selected, i.e. all balls drawn are of the same color after some random but finite time. If $\alpha < 1$, balls of both colors will be drawn at random with probability $1/2$. The larger α or the closer α is to zero, the faster these effects will be seen.

When $\alpha = 1$, the asymptotic behavior of the proportion of red balls drawn is determined by c . As c grows from zero to infinity, the limiting distribution evolves continuously from two Dirac point masses at 0 and 1 to a single Dirac mass at $1/2$. To illustrate this point, we show in Figure 1 the density of the Beta distribution for $c = 0.1$ and $c = 10$. We make some further comments.

- If $c < 1$, a strong asymmetry in the choices of the color appears. One branch is eventually chosen much more frequently than the other. Furthermore as c tends to 0, the Beta distribution tends to the distribution with two point masses at 0 and 1. This limit case corresponds to the situation in which a color is selected, i.e. $\alpha > 1$.
- If $c = 1$, the limiting distribution is uniform on $[0, 1]$.
- If $c > 1$, Z_n/n appears to be much more concentrated around $1/2$. This is similar to what is observed in the case $\alpha < 1$.

In this model, the first draws are greatly influential. If a red ball is first drawn, then the relative weight of red balls for the second draw is $\{(1+c)/c\}^\alpha$. Even if α is smaller than 1 but not too small, then a very small value of c will give an enormous weight to the red color, and thus the first few balls drawn are highly likely to be red. This might lead to the wrong conclusion that a color is selected, which means that $\alpha > 1$. On the other hand, if c is large, then even if α is larger than 1, the relative weight of red balls will not be large for the first few draws, and thus it will be difficult to see that a color is selected in the early stages of the experiment. Figure 2 illustrates these phase transitions.

In conclusion, we can expect that the estimation will be difficult when both parameters contribute to the same effect, e.g. α large and c small (fast selection of one color) or α small and c large (no selection); and also when the parameters have competing effects: very small c and $\alpha < 1$, or very large c and $\alpha > 1$. This will be illustrated in Section 3.4.

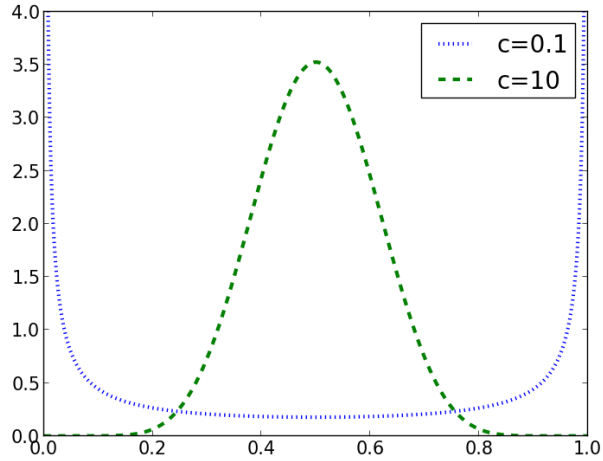


Figure 1: Graph of the Beta distribution for parameters (c, c) with $c = 0.1$ and $c = 10$

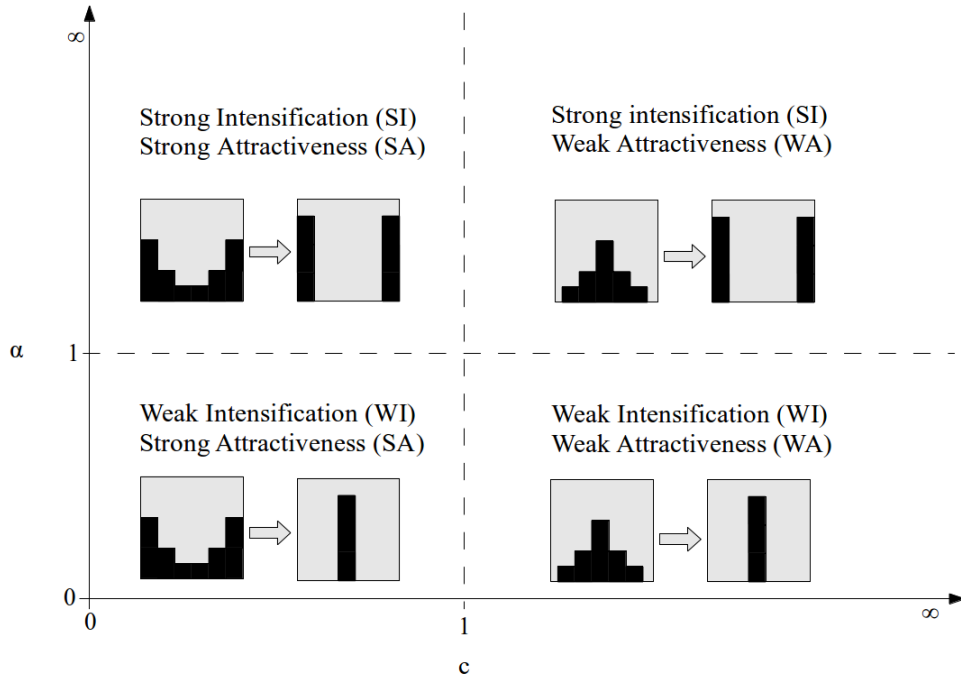


Figure 2: Phase diagram of the model. The graphs in the shaded boxes show the shape of the empirical distribution of Z_n/n for small n (left) and its limiting distribution (as $n \rightarrow \infty$, right).

3.1 Parametric estimation

We start with the maximum likelihood estimator, that is $\hat{\theta}_N = (\hat{\alpha}_N, \hat{c}_N)$ defined by (2.3). The Fisher information matrix has the following expression.

$$\mathcal{I}_n(\theta) = \sum_{k=0}^{n-1} \sum_{i=0}^k \mathbb{P}(Z_k = i) f(\alpha, c, i, k-i) \bar{f}(\alpha, c, i, k-i) \mathcal{J}(\alpha, c, i, k-i), \quad (3.4)$$

with $\theta = (\alpha, c)$ and for $0 \leq i \leq k \leq n-1$,

$$\mathcal{J}(\alpha, c, i, k-i) = \begin{pmatrix} \log^2 \left(\frac{c+i}{c+k-i} \right) & \log \left(\frac{c+i}{c+k-i} \right) \frac{\alpha(k-2i)}{(c+i)(c+k-i)} \\ \log \left(\frac{c+i}{c+k-i} \right) \frac{\alpha(k-2i)}{(c+i)(c+k-i)} & \frac{\alpha^2(k-2i)^2}{(c+i)^2(c+k-i)^2} \end{pmatrix}.$$

It is important to note that the Fisher information matrix is not diagonal. Thus the estimation of each parameter has an effect on the estimation of the other.

Theorem 3.2. *Let Θ be a compact subset of $(0, \infty)^2$ which contains (α_0, c_0) . Then the maximum likelihood estimator $\hat{\theta}_N$ is consistent and asymptotically Gaussian and efficient, i.e. $\sqrt{N}(\hat{\theta}_N - \theta_0)$ converges weakly to $\mathcal{N}(0, \mathcal{I}_n^{-1}(\theta_0))$.*

The proof is in Section 5.4. Since Assumption 2.1-(i) obviously holds, it remains only to check (ii)-(iii) of Assumption 2.1.

As mentioned above, we will use weighted least squared estimators defined by (2.8) for several different weight sequences w_N , such that $w_N(i, k-i)$ converges almost surely to $w_0(i, k-i) > 0$, for all $0 \leq k \leq n-1$ and $0 \leq i \leq k$. Their limiting covariance matrix is written:

$$\Sigma_n(\theta_0) = \ddot{W}^{-1}(\theta_0) H(\theta_0) \ddot{W}^{-1}(\theta_0)$$

with

$$H(\theta_0) = 4 \sum_{k=0}^{n-1} \sum_{i=0}^k w_0(i, k-i)^2 \frac{f_0^3(i, k-i) \bar{f}_0^3(i, k-i)}{\mathbb{P}(Z_k = i)} \mathcal{J}(\alpha_0, c_0, i, k-i)$$

and

$$\ddot{W}(\theta_0) = 2 \sum_{k=0}^{n-1} \sum_{i=0}^k w_0(i, k-i) f_0^2(i, k-i) \bar{f}_0^2(i, k-i) \mathcal{J}(\alpha_0, c_0, i, k-i).$$

Theorem 3.3. *Let Θ be a compact subset of $(0, \infty)^2$ which contains (α_0, c_0) and assume that w_N converges almost surely to positive weights w_0 . Then the weighted least squared estimator $\hat{\theta}_N^W$ is consistent and asymptotically Gaussian, i.e. $\sqrt{N}(\hat{\theta}_N^W - \theta_0)$ converges weakly to $\mathcal{N}(0, \Sigma_n(\theta_0))$. It is efficient if $w_N(i, k-i) = p_N^{-1} q_N^{-1} a_N(i, k-i)$, for all $0 \leq k \leq n-1$ and $0 \leq i \leq k$.*

3.2 Estimation of α as a tail index

If $\alpha > 1$, the limit (3.2) in Theorem 3.1 implies that the total number of balls drawn of the color eventually not selected has a tail index α :

$$\lim_{x \rightarrow \infty} x^{\alpha-1} \mathbb{P}(Q_\infty > x) = \frac{Q^*}{\alpha - 1}.$$

Therefore, a natural statistical idea is to estimate α as a tail index. Since this tail index depends only on α and not on c , it might appear at first sight as an interesting possibility.

However, in the present framework, we do not observe the total number of balls Q_∞ of the discarded color since we only observe N finite experiments of length n . For the j -th experiment, we estimate Q_∞ by $Q_n = \min(Z_n, n - Z_n)$. By definition, $Q_n \leq n/2$ and $Q_n \leq Q_\infty$. Applying Theorem 3.1, the probability of error can be bounded as follows:

$$\begin{aligned} \mathbb{P}(Q_n < Q_\infty) &= \mathbb{P}(Q_n < Q_\infty ; T_\infty > n) + \mathbb{P}(Q_n < Q_\infty ; T_\infty \leq n) \\ &= \mathbb{P}(T_\infty > n) + \mathbb{P}(Q_n < Q_\infty ; T_\infty \leq n) \\ &\leq \mathbb{P}(T_\infty > n) + \mathbb{P}(Q_\infty > n/2) = O(n^{-(\alpha+1/\alpha-2)}). \end{aligned}$$

The error will be large if α is very close to 1, but it appears from simulations that the error is actually much smaller than this bound for $\alpha > 1.5$. Therefore we have tried to estimate α as a tail index by pretending that we observe uncensored data, i.e. infinite paths. The best-known estimator of the tail index is the so-called Hill estimator [Res07]. We have tried to implement the Hill estimator as described in [DRDH00] but the simulation results were extremely poor and we do not report them. However, we think that this method might be useful since it has the advantage not to depend on the parameter c . It is the subject of further research to estimate efficiently α as a tail index.

3.3 Estimation on a single path

The main feature of the reinforced random walk model for $\alpha > 1$ is that balls of only one color will be drawn eventually. It seems then clear that a statistical procedure based on only one path cannot be consistent, since no new information will be obtained after one color stops appearing in the draws. This intuition is true and more surprisingly, it is also true in the case $\alpha = 1$. This is translated in statistical terms in the following theorem.

Let ℓ_n denote the log-likelihood based on a single path of length n and $\dot{\ell}_n$ its gradient. The model is regular, so the Fisher information is $\text{var}_\theta(\dot{\ell}_n(\theta))$.

Theorem 3.4.

- (i) If $\alpha = 1$ and $c > 0$, then $\lim_{n \rightarrow \infty} \mathcal{I}_n(c) < \infty$.
- (ii) If $\alpha < 1$, $n^{-1} \ell_n(\theta) \rightarrow -\log 2$.

(iii) If $\alpha > 1$, then $\ell_n(\theta)$ converges almost surely to a random variable as $n \rightarrow \infty$.

The proof is in Section 5.4. Statement (i) means that when $\alpha = 1$, the Fisher information is bounded. This implies that the parameter c cannot be estimated on a single path. This also implies that the length n of each path should be taken as large as possible (theoretically infinite) in order to minimize the asymptotic variance of the estimators. Statements (ii) and (iii) imply that the maximum likelihood estimator is inconsistent, since the likelihood does not tend to a constant.

3.4 Estimation on simulations

In order to assess the quality of the estimators proposed in Section 3.1, we have made a short simulation study. For several pairs (α, c) , we have simulated 1000 experiments of $N = 50$ paths of length $n = 100$ (recall that n is the number of balls drawn or number of ants going through the bifurcation). These are reasonable values in view of the practical experiments with actual ants. We compare the performance of the maximum likelihood estimator $\hat{\theta}_N$ (MLE) defined in (2.3) and of the weighted least squares estimator $\hat{\theta}_N^W$ (WLSE) defined in (2.8) with the weights $w_N(i, k-i) = a_N(i, k-i)$ defined in (2.5). The asymptotically efficient weighted least squares estimator with the weights $w_N(i, k-i) = a_N(i, k-i)p_N(i, k-i)^{-1}q_N(i, k-i)^{-1}$ provides a severely biased estimation of α and always estimates a very small value of c with a very small dispersion. This is caused by the fact that the empirical p_N and q_N vanish frequently, so that the weights are infinite. We will not report the simulations for this estimator.

The RRW model is symmetric in the sense that $f(\alpha, c, k-i, i) = 1 - f(\alpha, c, i, k-i)$. We will take an advantage of this property to improve the numerical efficiency of the proposed WLSE. If we define, for $0 \leq i \leq k \leq n-1$,

$$b_N(i, k-i) = p_N(i, k-i)a_N(i, k-i) = \frac{1}{N} \sum_{j=1}^N \mathbb{1}_{\{Z_k^j=i\}} X_{k+1}^j, \quad (3.5)$$

$$\tilde{p}_N(i, k-i) = \frac{b_N(i, k-i) + a_N(k-i, i) - b_N(k-i, i)}{a_N(i, k-i) + a_N(k-i, i)}, \quad (3.6)$$

then we can write (see Section 5.3)

$$\hat{\theta}_N^W = \arg \min_{\alpha, c} \sum_{k=1}^n \sum_{i=0}^{\lfloor k/2 \rfloor} (w_N(i, k-i) + w_N(k-i, i)) \{f(\alpha, c, i, k-i) - \tilde{p}_N(i, k-i)\}^2. \quad (3.7)$$

Similarly to $p_N(i, k-i)$, the empirical probability $\tilde{p}_N(i, k-i)$ is a consistent estimator of $f_0(i, k-i)$ and moreover uses twice as much data. In the simulations, we use this definition of the WLSE to estimate α and c .

In order to assess the quality of the estimators, we first evaluate numerically some values of the theoretical standard deviations of both estimators for several values of α and c . We have chosen arbitrary values of α and c in the range 0.5, 2. We have also chosen values of α and c which correspond to those found in the literature cited and to those that we have estimated in the real experiment described in section 3.5. These simulation results are reported in Table 1 and are plotted as functions of α for different values of c in Figures 3 and 4.

- As theoretically expected, the asymptotic variance of the MLE, which is the Fisher information bound, is smaller than the variance WLSE, but the ratio is never less than one fourth. Moreover, their overall behavior is similar.
- The variance of the estimators of α is smaller when both parameters do not contribute to the same effect. The worst variance is for α large and c small, that is when the values of both parameters imply fast selection of a color. The variance tend to infinity when α tends to infinity.
- The variance of the estimators of c increases with c and tends to infinity when α tends to 0 and to ∞ .
- These effects are explained by the fact that the coefficients of the Fisher information matrix tend to zero when α tends to zero, except the coefficient corresponding to α . See Formula (3.4).

(α, c)	$\sigma_{\alpha}^{MLE}/\sqrt{N}$	$\sigma_{\alpha}^{WLSE}/\sqrt{N}$	σ_c^{MLE}/\sqrt{N}	σ_c^{WLSE}/\sqrt{N}
(0.5, 0.5)	$5.02 \cdot 10^{-2}$	$6.25 \cdot 10^{-2}$	$2.54 \cdot 10^{-1}$	$3.17 \cdot 10^{-1}$
(0.5, 1.0)	$6.45 \cdot 10^{-2}$	$8.56 \cdot 10^{-2}$	$5.98 \cdot 10^{-1}$	$7.94 \cdot 10^{-1}$
(0.5, 2.0)	$8.80 \cdot 10^{-2}$	$1.29 \cdot 10^{-1}$	1.46	2.14
(1.0, 0.5)	$3.81 \cdot 10^{-2}$	$4.97 \cdot 10^{-2}$	$1.18 \cdot 10^{-1}$	$1.59 \cdot 10^{-1}$
(1.0, 1.0)	$4.34 \cdot 10^{-2}$	$5.91 \cdot 10^{-2}$	$2.52 \cdot 10^{-1}$	$3.49 \cdot 10^{-1}$
(1.0, 2.0)	$5.83 \cdot 10^{-2}$	$8.72 \cdot 10^{-2}$	$5.93 \cdot 10^{-1}$	$8.91 \cdot 10^{-1}$
(1.5, 0.5)	$7.83 \cdot 10^{-2}$	$1.28 \cdot 10^{-1}$	$1.23 \cdot 10^{-1}$	$1.87 \cdot 10^{-1}$
(1.5, 1.0)	$6.69 \cdot 10^{-2}$	$1.08 \cdot 10^{-1}$	$2.10 \cdot 10^{-1}$	$3.31 \cdot 10^{-1}$
(1.5, 2.0)	$6.88 \cdot 10^{-2}$	$1.15 \cdot 10^{-1}$	$4.17 \cdot 10^{-1}$	$6.88 \cdot 10^{-1}$
(2.0, 0.5)	$1.94 \cdot 10^{-1}$	$4.18 \cdot 10^{-1}$	$1.64 \cdot 10^{-1}$	$2.86 \cdot 10^{-1}$
(2.0, 1.0)	$1.35 \cdot 10^{-1}$	$2.87 \cdot 10^{-1}$	$2.39 \cdot 10^{-1}$	$4.43 \cdot 10^{-1}$
(2.0, 2.0)	$1.12 \cdot 10^{-1}$	$2.31 \cdot 10^{-1}$	$4.07 \cdot 10^{-1}$	$7.72 \cdot 10^{-1}$
(2.0, 20.0)	$3.55 \cdot 10^{-1}$	1.29	7.94	$2.88 \cdot 10^1$
(2.6, 60.0)	1.66	$1.23 \cdot 10^1$	$5.78 \cdot 10^1$	$4.32 \cdot 10^2$
(1.1, 3.0)	$7.20 \cdot 10^{-2}$	$1.19 \cdot 10^{-1}$	$8.99 \cdot 10^{-1}$	1.48
(1.1, 7.0)	$1.38 \cdot 10^{-1}$	$2.95 \cdot 10^{-1}$	2.97	6.33

Table 1: Theoretical standard deviation for $N = 50$ paths of length 100

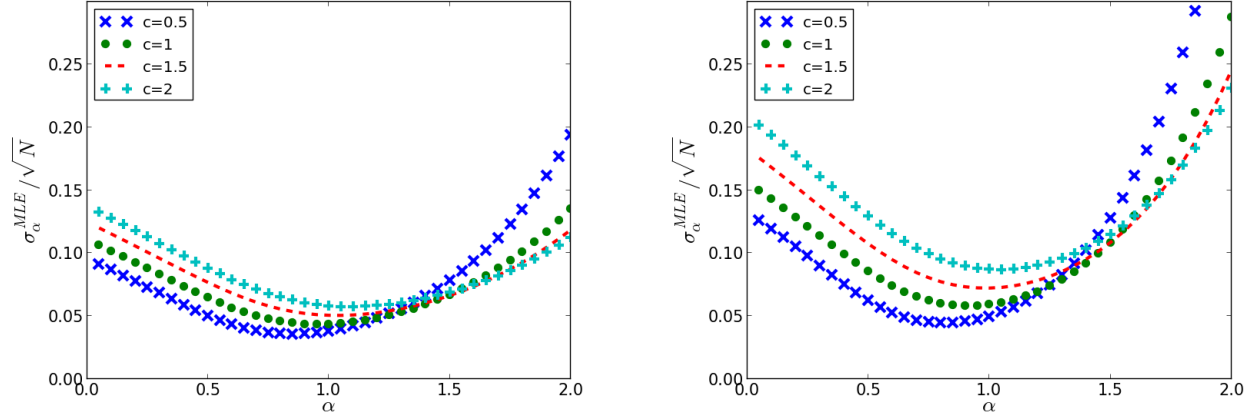


Figure 3: Theoretical standard deviation for $N = 50$ paths of length $n = 100$ of $\hat{\alpha}^{MLE}$ (left) and $\hat{\alpha}^{WLS}$ (right), for α in $(0, 2]$ and fixed values of c .

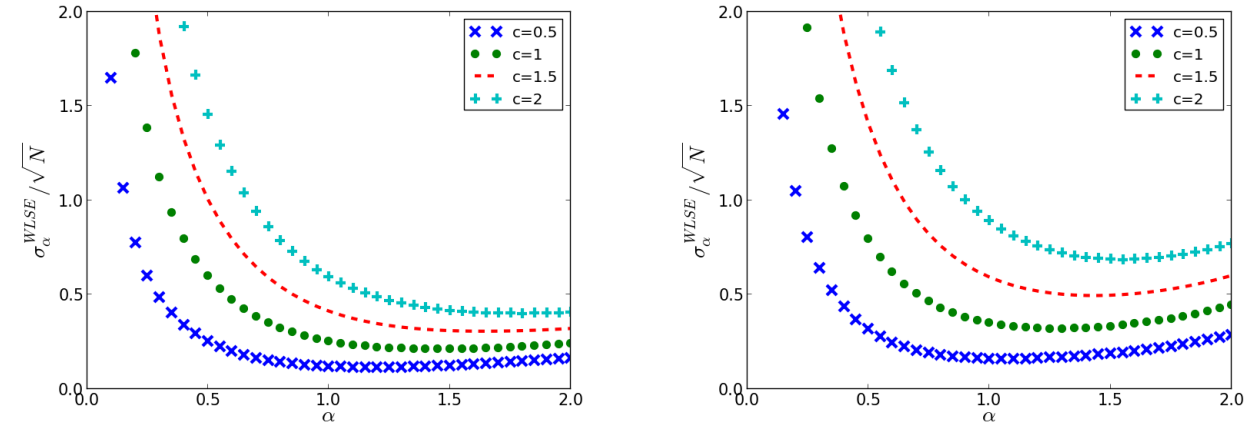


Figure 4: Theoretical standard deviation for $N = 50$ paths of length $n = 100$ of \hat{c}^{MLE} (left) and \hat{c}^{WLS} (right), for α in $(0, 2]$ and fixed values of c .

In Tables 2 and 3, we report the performance of the estimators on the simulated data for the same values of the parameters. Recall that we have simulated 1000 experiments, each of $N = 50$ paths of length $n = 100$. Because of the length of the computations, each MLE was computed only 500 times.

- Table 2 report the square root of the MSE of both estimators. They share the same feature as the theoretical standard deviations: they increase when both parameters concur to the same effect and when c is large. This increase is more noticeable for the WLS than for the MLE. The performance of the MLE is the poorest when both parameters concur to prevent selection of one color and when c is very large.

- The increase of the MSE for certain values of the parameters is in part due to the skewness of the estimators. For these values, both estimators tend to overestimate the parameters. Table 3 shows the 10% and 90% empirical quantiles of the estimators for some of the values of the parameters giving the worst results in Tables 1 and 2.
- As expected, both estimators have poorer performances when the parameters strongly concur to the same effect, either selection of a color or non selection. This can be seen as a problem of identifiability. For instance, the apparent selection of a color can lead in finite samples to a very small estimate of c and an estimate of α smaller than 1 as well as a moderate estimate of c and a value of α large than 1. Asymptotically, the theory says that the estimators will converge to the true values, but these competing effect of the parameters may explain a higher variability in finite samples. This loss of performance is much stronger for the WLSE than for the MLE and moreover. For the MLE it is much stronger in the case of non selection than in the case of selection where the empirical performance of the MLE nearly matches the theoretical value. These effects are always stronger for the estimation of c than for the estimation of α .
- In the case where selection of a color is fast, then many of the empirical weights used to compute the MLE vanish, and the least squares method uses very few points to fit the curve. The MLE is not affected by this problem.
- In the case where both parameter concur to non selection, then the probability of choosing one color converges very fast to $1/2$, and thus the experiments brings very little information. This affects both the MLE and the WLSE, and in addition, many of the empirical weights vanish so the WLSE is even less efficient.

An important feature is that the two estimators are asymptotically correlated, as appears in the asymptotic covariance matrix. Figure 5 shows a scatterplot of the 500 simulations of the MLE and WLSE for the true values $\alpha = 1.1$ et $c = 7$. In both cases, the estimate of α appears to be (on average) an increasing value of the estimated of c . Even though the value of α is very close to one, nearly 80% of the MLE estimates and 70% of the WLSE estimates of α are above the value 1.

Since the asymptotic variances depend on the unknown parameter, we have computed the pivotal Bootstrap 95% confidence intervals for the parameters, based on one simulation of $N = 50$ paths of length $n = 100$ and a Bootstrap sample size of 500. See [Was04, Section 8.3] for details on this method. We compare these Bootstrap intervals with the corresponding Monte-Carlo intervals, based on 500 simulations. These intervals are reported in Table 4. The match is nearly perfect for the MLE for α , but as before, the performance is poorer for the estimation of c . The intervals for c are noticeably skewed to the right but always contain the true value.

(α, c)	$\sqrt{MSE_{\alpha}^{MLE}}$	$\sqrt{MSE_{\alpha}^{WLSE}}$	$\sqrt{MSE_c^{MLE}}$	$\sqrt{MSE_c^{WLSE}}$
(0.5, 0.5)	$5.60 \cdot 10^{-2}$	$7.60 \cdot 10^{-2}$	$5.41 \cdot 10^{-1}$	$5.88 \cdot 10^{-1}$
(0.5, 1.0)	$7.39 \cdot 10^{-2}$	$1.02 \cdot 10^{-1}$	1.37	1.69
(0.5, 2.0)	$1.69 \cdot 10^{-1}$	3.56	6.51	$2.95 \cdot 10^2$
(1.0, 0.5)	$3.90 \cdot 10^{-2}$	$6.25 \cdot 10^{-2}$	$1.24 \cdot 10^{-1}$	$2.10 \cdot 10^{-1}$
(1.0, 1.0)	$4.98 \cdot 10^{-2}$	$7.34 \cdot 10^{-2}$	$3.13 \cdot 10^{-1}$	$4.73 \cdot 10^{-1}$
(1.0, 2.0)	$5.80 \cdot 10^{-2}$	$9.38 \cdot 10^{-2}$	$6.49 \cdot 10^{-1}$	1.11
(1.5, 0.5)	$7.85 \cdot 10^{-2}$	$1.94 \cdot 10^{-1}$	$1.30 \cdot 10^{-1}$	$2.46 \cdot 10^{-1}$
(1.5, 1.0)	$6.59 \cdot 10^{-2}$	$1.51 \cdot 10^{-1}$	$2.14 \cdot 10^{-1}$	$4.03 \cdot 10^{-1}$
(1.5, 2.0)	$7.24 \cdot 10^{-2}$	$1.39 \cdot 10^{-1}$	$4.59 \cdot 10^{-1}$	$7.84 \cdot 10^{-1}$
(2.0, 0.5)	$2.61 \cdot 10^{-1}$	$2.69 \cdot 10^1$	$1.98 \cdot 10^{-1}$	$1.41 \cdot 10^1$
(2.0, 1.0)	$1.42 \cdot 10^{-1}$	9.85	$2.60 \cdot 10^{-1}$	9.88
(2.0, 2.0)	$1.17 \cdot 10^{-1}$	$3.54 \cdot 10^{-1}$	$4.34 \cdot 10^{-1}$	1.01
(2.0, 20.0)	$5.88 \cdot 10^{-1}$	1.24	$1.42 \cdot 10^1$	$3.24 \cdot 10^1$
(2.6, 60.0)	8.64	$3.59 \cdot 10^1$	$3.25 \cdot 10^2$	$1.42 \cdot 10^3$
(1.1, 3.0)	$7.80 \cdot 10^{-2}$	$1.13 \cdot 10^{-1}$	1.11	2.01
(1.1, 7.0)	$1.74 \cdot 10^{-1}$	$2.83 \cdot 10^{-1}$	4.33	7.58

Table 2: Square root of the mean square error (MSE) for 500 (for the MLE) or 1000 (for the WLSE) simulated experiences of $N = 50$ paths of length $n = 100$.

	(α, c)	$(D_1^{\alpha}, D_9^{\alpha})$	(D_1^c, D_9^c)
MLE	(0.5, 2.0)	(0.39, 0.65)	(0.83, 5.65)
	(2.0, 0.5)	(1.79, 2.35)	(0.33, 0.82)
WLSE	(0.5, 2.0)	(0.34, 0.70)	(0.45, 8.29)
	(2.0, 0.5)	(1.60, 3.73)	(0.24, 1.57)

Table 3: Empirical deciles for 500 (for the MLE) or 1000 (for the WLSE) simulated experiences of $N = 50$ paths of length $n = 100$.

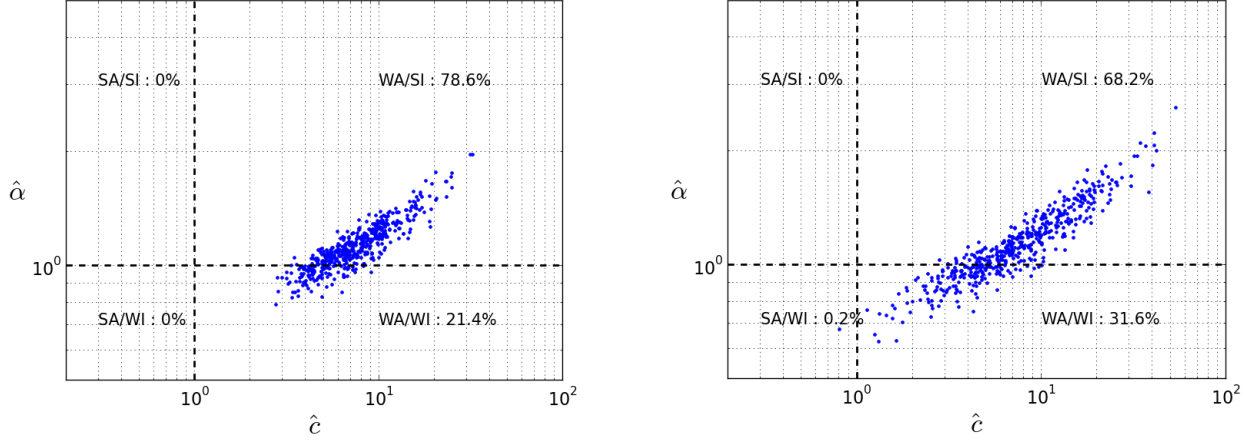


Figure 5: Scatterplots of the estimates $(\hat{\alpha}, \hat{c})$ for the 500 simulated experiences of $N = 50$ paths of length $n = 100$.

	(α, c)	IDC 95% for α		IDC 95% for c	
		Monte-Carlo	Bootstrap	Monte-Carlo	Bootstrap
MLE	(1.0, 1.0)	(0.91, 1.11)	(0.91, 1.09)	(0.62, 1.88)	(0.75, 1.82)
	(1.0, 2.0)	(0.90, 1.13)	(0.92, 1.17)	(1.20, 3.71)	(1.17, 3.70)
	(1.5, 1.0)	(1.38, 1.65)	(1.34, 1.57)	(0.68, 1.51)	(0.42, 1.07)
	(1.5, 2.0)	(1.38, 1.66)	(1.42, 1.74)	(1.39, 3.06)	(1.55, 3.63)
-----	(2.0, 20)	(1.50, 3.38)	(1.90, 5.10)	(9.90, 55.5)	(19.1, 94.4)
	(2.6, 60)	(1.23, 29.3)	(1.12, 38.7)	(15.9, 1053)	(19.9, 1637)
	(1.1, 3.0)	(0.98, 1.29)	(1.01, 1.23)	(1.81, 6.22)	(1.34, 3.99)
	(1.1, 7.0)	(0.89, 1.53)	(0.75, 1.44)	(3.5, 19.16)	(3.05, 19.8)
WLSE	(1.0, 1.0)	(0.87, 1.16)	(0.89, 1.14)	(0.49, 2.17)	(0.57, 2.07)
	(1.0, 2.0)	(0.84, 1.22)	(0.80, 1.18)	(0.87, 5.13)	(0.56, 3.49)
	(1.5, 1.0)	(1.26, 1.83)	(1.24, 1.64)	(0.47, 2.02)	(0.29, 1.20)
	(1.5, 2.0)	(1.26, 2.26)	(1.41, 2.10)	(0.94, 3.82)	(1.71, 6.70)
-----	(2.0, 20)	(1.26, 4.85)	(1.06, 3.99)	(5.92, 88.1)	(2.76, 69.3)
	(2.6, 60)	(0.74, 88.3)	(0.24, 87.7)	(3.86, 3754)	(0.17, 4132)
	(1.1, 3.0)	(0.89, 1.41)	(1.06, 1.44)	(1.21, 8.72)	(1.70, 6.76)
	(1.1, 7.0)	(0.75, 1.83)	(0.55, 2.56)	(1.68, 30.3)	(1.48, 66.0)

Table 4: Monte-Carlo 95% confidence intervals for 500 simulated experiment of $N = 50$ paths of length $n = 100$ and Bootstrap 95% confidence intervals for one simulated experiment of 50 paths of length 100

3.5 Ants: estimation on life experience

In this section, we will compute the previous estimators on the results of a path selection experiment by a colony of ants. This experiment has been done in 50 replicas in the Research Center on Animal Cognition (UMR 5169) of Paul Sabatier University Toulouse under the supervision of Guy Theraulaz, Hugues Chaté and the first author. A small laboratory colony (approximately 200 workers) of Argentine ants *Linepithema humile* has been starved for two days before the experiment. During the experiment, the colony has access to a fork carved in a white PVC slab and partially covered by a Plexiglas plate (see Figure 6). The angle between the branches is 60° . The fork galleries have a 0.4 cm square section. The entrance of the maze is controlled by a door. Food is never present during the experiment.

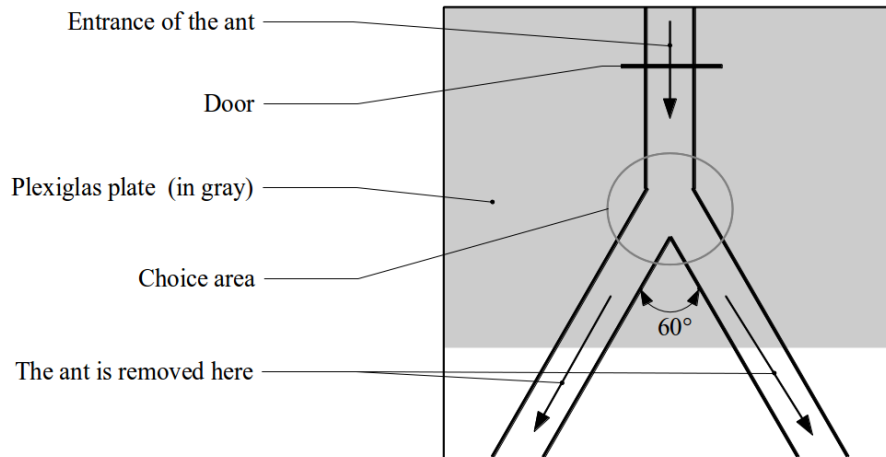


Figure 6: The experimental set-up: a fork carved in a white PVC slab, partially covered by a Plexiglass plate.

The experiment consists in introducing a single ant in the set-up. Once inside, the only possibility for the ant to go forward is to choose between the left or the right branch of the fork. As soon as the ant makes a choice and steps into one branch, it is removed from the set-up and another ant is introduced. All the choices are noted and the experiment ends, when 100 ants have passed through the fork.

There is never more than one ant in the set-up. This implies that the ant in the maze receives no other cue about the previous passages than the pheromone that has been laid.

Figure 7 shows the 50 paths of length $n = 100$, that is 100 ants went through the bifurcation. The paths are represented as random walks with increment $+1$ when the right branch is chosen, and -1 when the left one is chosen. In less than ten experiments, a branch is clearly selected, whereas in the others, selection of a branch is not obvious. Figure 8 shows the histogram of the distribution of $Z_{100}/100$, that is the final proportion of the choices of the right branch. There is no clear visual evidence that $\alpha > 1$.

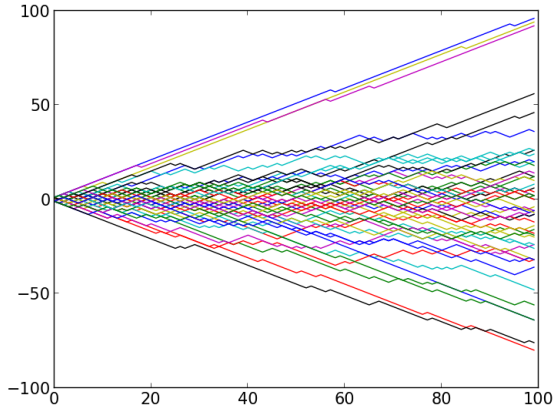


Figure 7: The 50 paths of $n = 100$ ants choosing either left ($+1$) or right (-1)

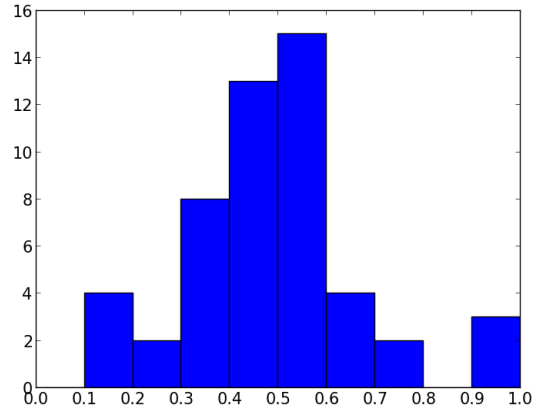


Figure 8: Histogram of the final proportion of right passages ($Z_{100}/100$)

Several values of these parameters have been proposed in the applied literature. ([DAGP90]) proposed $\alpha = 2$, $c = 20$ and more recently ([GGC⁺09]) suggested $\alpha = 2.6$ and $c = 60$. It must be noted however that these values are not obtained by a statistical method but by the calibration of a curve to a plot. Therefore these methods do not lead to confidence intervals. Moreover, a calibration method has an inherent risk of over fitting. As illustrated in Figure 2, if for instance α and c are both small, then both branches will be asymptotically equally chosen, but paths of finite length n might be misleading and the calibration will suggest values of α and c corresponding to the selection of a branch. The statistical procedure is based on a model for the dynamics of the process and is thus less prone to this type of error. Nevertheless, we will see that our results do not contradict those of [DAGP90] and [GGC⁺09], but complement them.

Table 5 shows the results of the ML and WLS estimations. Both estimates of α are close to 1.1 and the estimates of c are between 3 and 7. The 95% confidence intervals are slightly larger than the simulated ones. This increased variability is due to the extreme paths which seem to show a very fast selection of one branch (see Figures 7 and 8). This may suggest that the ants did not have the same behavior and that the distribution of $Z_{100}/100$ could be a mixture of two distributions.

For both methods, the 95% Bootstrap confidence intervals of α contain the value 1. More precisely, as shown in Figure 9, approximately 1/3 of the bootstrap parameters gives weak Attractiveness ($c > 1$) and weak Intensification ($\alpha < 1$), which means that no branch is eventually selected. In almost all the others cases, we conclude to weak Attractiveness

($c > 1$) and strong Intensification ($\alpha > 1$), which means that a branch will be eventually, though slowly, selected. Only in a few cases the estimators gives strong Attractiveness ($c < 1$), but a weak Intensification ($\alpha < 1$), which means that a branched is much more choosen than the other at the beginning of the experiment. Finally, there are no values which imply both strong Attractiveness ($c < 1$) and Intensification ($\alpha > 1$). Therefore we can conclude that the Attractiveness is weak, i.e. $c > 1$, and that the Intensification is strong $\alpha > 1$, though not with an extremely great confidence in the latter case.

The values obtained by [DAGP90] ($\alpha = 2$, $c = 20$) and more recently by [GGC⁺09] ($\alpha = 2.6$, $c = 60$) are both in the confidence intervals for the WLSE found in table 5. But the values of α suggested by these authors are out of the 95% confidence interval for the MLE. This confirms the idea that the calibration methods used in these references are closer to the weighted least squares method. Thus these parameters, which decides for a slow selection branch, are no much more likely than a parameter set, which yields a non selection path.

Similarly to Figure 5, Figure 9 confirms the fact that the two estimators parameters are strongly correlated and that $\hat{\alpha}^*$ is on average an increasing function of \hat{c}^* . There seems to be two cutoff values for c : if $\hat{c}^* > 8$, then $\hat{\alpha}^* > 1$, and if $\hat{c}^* < 1.5$, then $\hat{\alpha}^* < 1$. The above mentioned values reported by [DAGP90] and [GGC⁺09] exhibit these features: they both have $c > 8$ and $\alpha > 1$ and α increase with c .

If we fix the value of c and estimate only α , then the 95% Bootstrap confidence intervals for α are smaller. Figure 10 shows the estimated values of α and the confidence intervals as functions of the fixed value of c . We see that if c is greater than 6 for the MLE and than 12 for the WLSE, then the confidence intervals of α lie entirely above 1.

The last remarks show that if the attractiveness is weak enough, i.e. $c > 6$, we can conclude that selection of a branch will occur with probability 1. Since $1/c$ can be interpreted as the amount of trail pheromone deposits by each ant, if c could be measured precisely in a biological experiment, then the estimation of α would be much easier.

	$\hat{\alpha}$	Bootstrap 95% CI	\hat{c}	Bootstrap 95% CI
MLE	1.07	(0.80, 1.99)	3.26	(1.14, 23.0)
WLSE	1.10	(0.62, 3.81)	6.91	(0.94, 85.4)

Table 5: The MLE and the WLSE for the 50 paths of real ants and their Bootstrap 95% confidence intervals

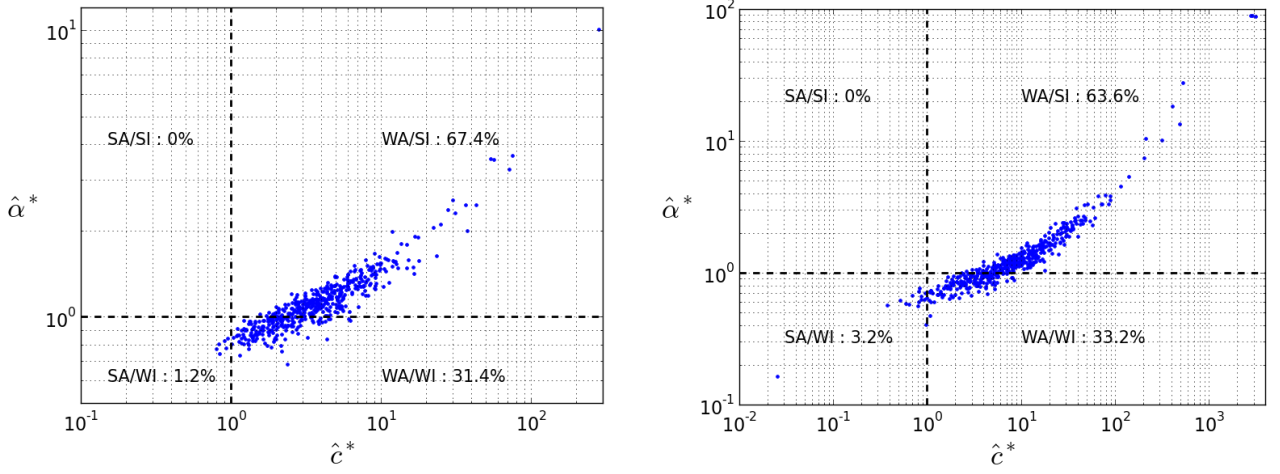


Figure 9: Log-log scatterplots of the estimates $(\hat{\alpha}^*, \hat{c}^*)$ for the 500 Bootstrap samples for the MLE (left) and the WLSE (right) for the 50 paths of real ants.

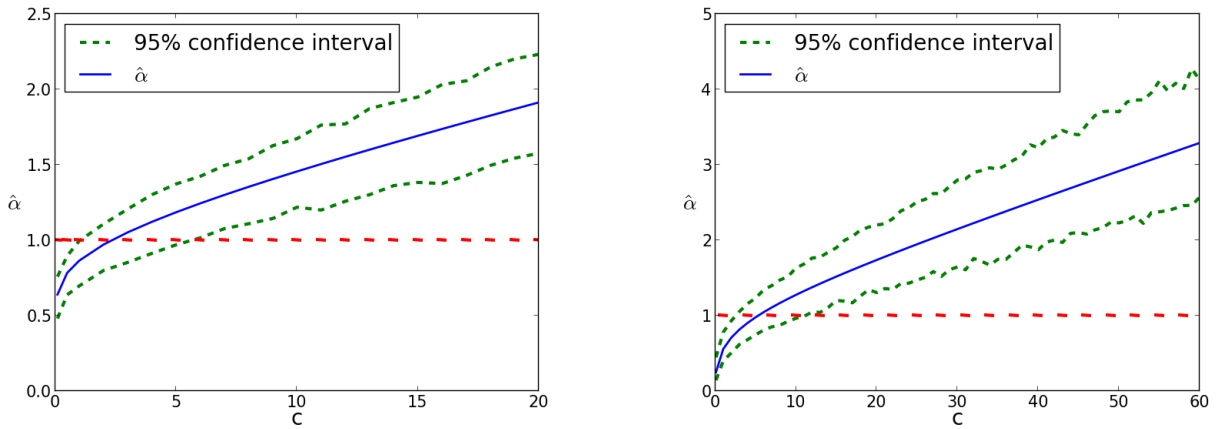


Figure 10: Graph of the estimates $\hat{\alpha}$ and their Bootstrap 95% confidence interval of the 50 paths of real ants as a function of fixed value of c

4 Concluding remarks

This article has presented two estimators of the parameters of a reinforced random walk model and possible generalizations. These estimators have been applied to data obtained in an experiment designed to investigate the path selection mechanism of the Argentine ant.

The reinforced random walk model (3.1) has two parameters α and c . The parameter α solely determines the behavior of the random walker deciding between left and right: if $\alpha > 1$, the walker will eventually always make the same choice. If $\alpha \leq 1$, then no selection happens. However, this result is asymptotic, and in finite paths, the value of c will have an important influence: if c is small, then the first choice will have an important and lasting influence on the next ones. This implies that the estimation of these parameters will be difficult since the same finite sample may come from two contradictory sets of parameters.

Our simulation results reported in Section 3.4 confirm this intuition and show that the maximum likelihood estimator performs well. We have also studied a least squares estimator which is (loosely) related to some of the methods of calibration of the parameters found in the biological literature.

We have then applied the proposed estimators to the Argentine ants experiment. In this context, the parameter c has a clear biological meaning: it is the inverse of the quantity of trail pheromone laid by each ant. The parameter α is not a physiological parameter. It can be interpreted as the intensification of the attractiveness of the pheromone. In the biological literature, values of $\alpha > 1$ have been reported, together with large values of c . This implies that the ants will select a branch with probability one, but the effect of a large c will be to delay this selection.

Our results confirm the biological findings, with some caveats. The estimated value of α is greater than one, but the 95% Bootstrap confidence intervals contain values smaller than one. Because of the strong correlation of the estimates, if c is constrained to large values (such as those found in the applied literature), then the confidence intervals of $\hat{\alpha}$ lie entirely above one. In conclusion, a better biological knowledge of c is needed in order to improve the statistical inference on α .

An important issue is that the reinforced random walk model does not take into account at all that the quantity of pheromone laid by the ants is not deterministic. All ants do not lay the same quantity of pheromone, and some ants do not even lay any pheromone. Also even in the experimental set-up, the pheromone evaporates at a certain (unknown) rate. Moreover, c also depends on the set-up: the angle between the branches, the material used, etc ... A direction of improvement of the model could be to allow c to be random. There is currently no theoretical result on such models.

In addition to this issue, further directions of research include the study of more general set-ups and models. Ants in nature do not only make binary choices. They may have more than two directions to choose, they have other sources of information and they may go

backwards. An experiment currently investigated at the CRCA (Centre de Recherches sur la Cognition Animale) laboratory in Toulouse consists in observing ants in a predefined, closed maze with several bifurcations. The statistical tools developed in this paper will be used in the analysis of this experiment.

All these behaviors have yet to be included in tractable models for which statistical inference can be made rigorously. This is the subject of current and further research.

5 Proofs

5.1 A central limit theorem for the empirical conditional probabilities

For $0 \leq i \leq k \leq n-1$, recall the definition of $a_N(i, k-i)$, $b_N(i, k-i)$ and $p_N(i, k-i)$ in (2.5) and in (3.5) and that $\bar{f}(\theta, i, k-i) = 1 - f(\theta, i, k-i)$.

Lemma 5.1. $\{\sqrt{N}(p_N(i, k-i) - f_0(i, k-i)), 0 \leq i \leq k \leq n-1\}$ converges weakly to a Gaussian vector with diagonal covariance matrix Γ_0 with diagonal elements

$$\Gamma_0(i, k-i) = \frac{f_0(i, k-i)\bar{f}_0(i, k-i)}{\mathbb{P}(Z_k = i)}. \quad (5.1)$$

The fact that the covariance matrix is diagonal is due to the Markovian structure.

Proof. Define $b_N(i, k-i) = N^{-1} \sum_{j=1}^N \mathbf{1}_{\{Z_k^j=i\}} X_{k+1}^j$, the empirical estimate of $b(i, k-i) = \mathbb{P}(Z_k = i, X_{k+1} = 1)$ and $a(i, k) = \mathbb{E}[a_N(i, k)] = \mathbb{P}(Z_k = i)$. Write then

$$\begin{aligned} p_N(i, k-i) - f_0(i, k-i) &= \frac{b_N(i, k-i) - b(i, k-i)}{a_N(i, k-i)} \\ &\quad - \frac{b(i, k-i)}{a_N(i, k-i)a(i, k-i)} (a_N(i, k-i) - a(i, k-i)). \end{aligned}$$

Since the paths (Z_1^j, \dots, Z_n^j) , $1 \leq j \leq N$ are i.i.d., the multivariate central limit holds for the sequence of $2n(n-1)$ dimensional vectors $\{(b_N(i, k-i) - b(i, k-i), a_N(i, k-i) - a(i, k-i)), 0 \leq i \leq k \leq n-1\}$. The proof is concluded by tedious computations using the Markov property which we omit. \square

Remark 5.2. We can prove that the covariance matrix Γ_0 is diagonal by a statistical argument. If we consider the tautological model $\{f(i, k-i), 0 \leq i \leq k \leq n-1\}$, i.e. $\theta = f$ and f_0 is the true value. Then the likelihood is

$$L_N(f) = \sum_{k=0}^{n-1} \sum_{i=0}^k a_N(i, k-i) \{p_N(i, k-i) \log f(i, k-i) + q_N(i, k-i) \log \bar{f}(i, k-i)\},$$

where $\bar{f}(i, k-i) = 1 - f(i, k-i)$. Thus we see that $\{p_N(i, k-i), 0 \leq i \leq k \leq n-1\}$ is the maximum likelihood estimator of f_0 . This model is a regular statistical model thus $\sqrt{N}(p_N - f_0)$ converges weakly to the Gaussian distribution with covariance matrix $I_n^{-1}(f_0)$, where $I_n(f)$ is the Fisher information matrix of the model. It is easily seen that $I_n(f_0)$ is the $n(n-1)$ dimensional diagonal matrix with diagonal elements given by (5.1).

5.2 A general result for minimum contrast estimators

Let $w_N(i, k-i)$, $0 \leq i \leq k \leq n-1$ be a sequence of random weights and let G be function defined on $[0, 1] \times (0, 1)$. Define the empirical contrast function by

$$\mathbb{W}_N(\theta) = \sum_{k=0}^{n-1} \sum_{i=0}^k w_N(i, k-i) G(p_N(i, k-i), f(\theta, i, k-i)) .$$

For instance, choosing $G(p, q) = -p \log q - (1-p) \log(1-q)$ and $w_N(i, k-i) = a_N(i, k-i)$ yields

$$\begin{aligned} \mathbb{W}_N(\theta) &= - \sum_{k=0}^{n-1} \sum_{i=0}^k a_N(i, k-i) \{p_N(i, k-i) \log f(\theta, i, k-i) + q_N(i, k-i) \log \bar{f}(\theta, i, k-i)\} \\ &= -N^{-1} L_N(\theta) , \end{aligned}$$

so that minimizing \mathbb{W}_n is equivalent to maximizing the likelihood. Choosing $G(p, q) = (p - q)^2$ yields the weighted least squares contrast function. We now define the minimum contrast estimator of θ_0 by

$$\hat{\theta}_N^{\mathbb{W}} = \arg \min_{\theta \in \Theta} \mathbb{W}_N(\theta) .$$

In order to prove the consistency and asymptotic normality of $\hat{\theta}_N^{\mathbb{W}}$, we make the following assumptions on G and on the weights $w_N(i, k-i)$. Let $\partial_2 G$ and $\partial_2^2 G$ denote the first and second derivatives of G with respect to its second argument.

Assumption 5.3. *The function G is non negative, twice continuously differentiable on $[0, 1] \times (0, 1)$ with $G(p, q) - G(p, p) > 0$ if $p \neq q$, $\partial_2 G(p, p) = 0$ and $\partial_2^2 G(p, p) > 0$.*

Assumption 5.4. *For all $0 \leq i \leq k \leq n-1$, $w_N(i, k-i)$ converge almost surely to $w_0(i, k-i)$ and $w_0(i, k-i) > 0$.*

Theorem 5.5. *Let Assumptions 2.1-(i) and (ii), 5.3 and 5.4 hold. Then $\hat{\theta}_N^{\mathbb{W}}$ is consistent. If moreover θ_0 is an interior point of Θ and Assumption 2.1-(iii) holds, then $\sqrt{N}(\hat{\theta}_N^{\mathbb{W}} - \theta_0)$ converges weakly to a Gaussian distribution with zero mean.*

The exact expression of the variance will be given in the proof.

Proof. Under Assumption 5.3, $\mathbb{W}_N(\theta)$ converges almost surely to

$$\mathbb{W}(\theta) = \sum_{k=0}^{n-1} \sum_{i=0}^k w_0(i, k-i) G(f_0(i, k-i), f(\theta, i, k-i)) .$$

Assumptions 2.1-(ii) and 5.3 ensure that θ_0 is the unique minimum of \mathbb{W} . Indeed, $G(p, q) > 0$ if $p \neq q$ and $G(p, p) = 0$. Thus, \mathbb{W} is minimized by any value of θ such that $f(\theta, i, k-i) = f(\theta_0, i, k-i)$. By Assumption 2.1-(ii), this implies $\theta = \theta_0$.

Assumptions 2.1 and 5.3 also imply that the convergence is uniform. This yields the consistency of $\hat{\theta}_N^{\mathbb{W}}$. For the sake of completeness, we give a brief proof. Since θ_0 minimizes \mathbb{W} and $\hat{\theta}_N^{\mathbb{W}}$ minimizes \mathbb{W}_N , we have:

$$\begin{aligned} 0 &\leq \mathbb{W}(\hat{\theta}_N^{\mathbb{W}}) - \mathbb{W}(\theta_0) \\ &= \mathbb{W}(\hat{\theta}_N^{\mathbb{W}}) - \mathbb{W}_N(\hat{\theta}_N^{\mathbb{W}}) + \mathbb{W}_N(\hat{\theta}_N^{\mathbb{W}}) - \mathbb{W}_N(\theta_0) + \mathbb{W}_N(\theta_0) - \mathbb{W}(\theta_0) \\ &\leq \mathbb{W}(\hat{\theta}_N^{\mathbb{W}}) - \mathbb{W}_N(\hat{\theta}_N^{\mathbb{W}}) + \mathbb{W}_N(\theta_0) - \mathbb{W}(\theta_0) \leq 2 \sup_{\theta \in \Theta} |\mathbb{W}_N(\theta) - \mathbb{W}(\theta)| . \end{aligned}$$

Since θ_0 is the unique minimizer of \mathbb{W} , for $\epsilon > 0$, we can find δ such that if $\theta \in \Theta$ and $\|\theta - \theta_0\| > \epsilon$, then $\mathbb{W}(\theta) - \mathbb{W}(\theta_0) \geq \delta$. Thus

$$\mathbb{P}(\|\hat{\theta}_N - \theta_0\| > \epsilon) \leq \mathbb{P}(\mathbb{W}(\hat{\theta}_N) - \mathbb{W}(\theta_0) \geq \delta) \leq \mathbb{P}\left(2 \sup_{\theta \in \Theta} |\mathbb{W}_N(\theta) - \mathbb{W}(\theta)| \geq \delta\right) \rightarrow 0 .$$

The central limit theorem is a consequence of the consistency and Lemma 5.1. A first order Taylor extension of $\dot{\mathbb{W}}_N(\theta)$ at θ_0 yields

$$0 = \dot{\mathbb{W}}_N(\hat{\theta}_N^{\mathbb{W}}) = \dot{\mathbb{W}}_N(\theta_0) + \ddot{\mathbb{W}}_N(\tilde{\theta}_N)(\hat{\theta}_N^{\mathbb{W}} - \theta_0) ,$$

with $\tilde{\theta}_N \in [\theta_0, \hat{\theta}_N^{\mathbb{W}}]$. Setting $\dot{f}_0(i, k-i) = \dot{f}(\theta_0, i, k-i)$, we have

$$\dot{\mathbb{W}}_N(\theta_0) = \sum_{k=0}^{n-1} \sum_{i=0}^k w_N(i, k-i) \partial_2 G(p_N(i, k-i), f_0(i, k-i)) \dot{f}_0(i, k-i) .$$

Let $\partial_{12}^2 G$ be the mixed second derivative of G . Note that $\partial_2 G(f_0(i, k-i), f_0(i, k-i)) = 0$, thus, by the delta-method (Cf.[DCD86, Theorem 3.3.11]),

$$\sqrt{N} \dot{\mathbb{W}}_N(\theta_0) \xrightarrow{w} \sum_{k=0}^{n-1} \sum_{i=0}^k w_0(i, k-i) \partial_{12}^2 G(f_0(i, k-i), f_0(i, k-i)) \Lambda_0(i, k-i) \dot{f}_0(i, k-i) .$$

where $\Lambda_0(i, k)$ are independent Gaussian random variables with zero mean and variance $\Gamma_0(i, k)$ defined by 5.1. Equivalently, $\sqrt{N} \dot{\mathbb{W}}_N(\theta_0)$ converges weakly to a Gaussian vector with zero mean and covariance matrix $H(\theta_0)$ defined by

$$H(\theta_0) = \sum_{k=0}^{n-1} \sum_{i=0}^k w_0^2(i, k-i) \{ \partial_{12}^2 G(f_0(i, k-i), f_0(i, k-i)) \}^2 \Gamma_0(i, k) \dot{f}_0(i, k-i) (\dot{f}_0(i, k-i))' .$$

By the law of large numbers, $\ddot{\mathbb{W}}_N(\theta)$ converges almost surely to $\ddot{\mathbb{W}}(\theta)$ and this convergence is also locally uniform. Thus, $\ddot{\mathbb{W}}_N(\tilde{\theta}_N)$ converges almost surely to $\ddot{\mathbb{W}}(\theta_0)$. Using again the fact that $\partial_2 G(p, p) = 0$, we obtain

$$\ddot{\mathbb{W}}(\theta_0) = \sum_{k=0}^{n-1} \sum_{i=0}^k w_0(i, k-i) \partial_2^2 G(f_0(i, k-i), f_0(i, k-i)) \dot{f}_0(i, k-i) (\dot{f}_0(i, k-i))' .$$

Denote for brevity $g(i, k-i) = w_0(i, k-i) \partial_2^2 G(f_0(i, k-i), f_0(i, k-i))$. Then, for any $u \in \mathbb{R}^d$, we have

$$u \ddot{\mathbb{W}}(\theta_0) u' = \sum_{k=0}^{n-1} \sum_{i=0}^k g(i, k-i) \left(\sum_{s=1}^d u_s \partial_s f(\theta_0, i, k-i) \right)^2 .$$

Since $g(i, k-i) > 0$ for all $0 \leq i \leq k \leq n-1$, this quantity can be zero only if for all $k = 0, \dots, n-1$ and $i = 0, \dots, k$,

$$\sum_{s=1}^d u_s \partial_s f(\theta_0, i, k) = 0 ,$$

By Assumption 2.1 (iii), this is possible only if $u_s = 0$ for all $s = 1, \dots, d$. Thus $\ddot{\mathbb{W}}(\theta_0)$ is positive definite.

We can now conclude that for large enough N , $\ddot{\mathbb{W}}_N(\tilde{\theta}_N)$ is invertible and we can write

$$\sqrt{N}(\hat{\theta}_N^{\mathbb{W}} - \theta_0) = -\ddot{\mathbb{W}}_N^{-1}(\tilde{\theta}_N) \sqrt{N} \dot{\mathbb{W}}_N(\theta_0) .$$

The right hand side converges weakly to the Gaussian distribution with zero mean and covariance matrix $\ddot{\mathbb{W}}^{-1}(\theta_0) H(\theta_0) \ddot{\mathbb{W}}^{-1}(\theta_0)$. \square

We can now prove Theorems 2.2 and 2.3.

Lemma 5.6. *Assumption 5.4 holds for the weights $a_N(i, k-i)$ and $a_N(i, k-i) p_N^{-1}(i, k-i) q_N^{-1}(i, k-i)$, $0 \leq i \leq k \leq n-1$.*

Proof. For all $0 \leq i \leq k \leq n-1$, the quantity $a_N(i, k-i)$ converges almost surely to $\mathbb{P}(Z_k = i)$ and $a_N(i, k-i) p_N^{-1}(i, k-i) q_N^{-1}(i, k-i)$ to $\mathbb{P}(Z_k = i) f_0(i, k-i)^{-1} \bar{f}_0(i, k-i)^{-1}$. Moreover Assumption 2.1 implies that $f_0(i, k-i) > 0$ and $\bar{f}_0(i, k-i) > 0$ for all $0 \leq i \leq k \leq n-1$. Using Formula (2.2), this in turn implies that $\mathbb{P}(Z_k = i) > 0$ for all $0 \leq i \leq k \leq n-1$. \square

Proof of Theorem 2.2. As mentioned above, the maximum likelihood estimator minimizes the contrast function \mathbb{W} obtained with the weights $a_N(i, k-i)$ and the function $G(p, q) = -p \log q - (1-p) \log(1-q)$. Thus the proof of Theorem 2.2 consists in checking Assumptions 5.3 and 5.4. Lemma 5.6 implies that Assumption 5.4 holds.

The function G considered here satisfies Assumption 5.3. Indeed, for $p, q \in (0, 1)$, define $K(p, q) = G(p, q) - G(p, p) = p \log(p/q) + (1 - p) \log((1 - p)/(1 - q))$, that is, $K(p, q)$ is the Kullback-Leibler distance between the Bernoulli measures with respective success probabilities p and q . It is well known that $K(p, q) > 0$ except if $p = q$. Indeed, by Jensen's inequality,

$$K(p, q) \geq -\log(pq/p + (1 - p)(1 - q)/(1 - p)) = \log 1 = 0 ,$$

and by strict concavity of the log function, equality holds only if $p = q$. Moreover, $\partial_2 G(p, q) = -p/q + (1 - p)/(1 - q)$ so $\partial_2 G(p, p) = 0$ and $\partial_2^2 G(p, p) = p^{-1}(1 - p)^{-1} > 0$. \square

Proof of Theorem 2.3. Again, the proof consists in checking Assumptions 5.3 and 5.4. The latter holds by virtue of Lemma 5.6 and Assumption 5.3 trivially holds for the function $G(p, q) = (p - q)^2$. \square

If $w_N(i, k - i) = p_N^{-1}(i, k - i)q_N^{-1}(i, k - i)a_N(i, k - i)$, then

$$H(\theta_0) = 2\ddot{W}(\theta_0) = 4\mathcal{I}_n(\theta_0) = 4 \sum_{k=0}^{n-1} \sum_{i=0}^k \frac{\mathbb{P}(Z_k = i)}{f_0(i, k - i)\bar{f}_0(i, k - i)} \dot{f}_0(i, k - i)(\dot{f}_0(i, k - i))' . \quad (5.2)$$

If the weights are chosen as $w_N(i, k) = a_N(i, k)$, then $w_0(i, k - i) = \mathbb{P}(Z_k = i)$ and

$$H(\theta_0) = 4 \sum_{k=0}^{n-1} \sum_{i=0}^k \mathbb{P}(Z_k = i) f_0(i, k - i) \bar{f}_0(i, k - i) \dot{f}_0(i, k - i) (\dot{f}_0(i, k - i))' , \quad (5.3)$$

$$\ddot{W}(\theta_0) = 2 \sum_{k=0}^{n-1} \sum_{i=0}^k \mathbb{P}(Z_k = i) \dot{f}_0(i, k - i) (\dot{f}_0(i, k - i))' . \quad (5.4)$$

5.3 Proof of the Equation (3.7)

Recall that the definitions of a_N , p_N , b_N and \tilde{p}_N in (2.5), (3.5) and (3.6). Denote $\lfloor \cdot \rfloor$ the floor function. Remark that:

$$a_N(i, k - i) - b_N(i, k - i) = \frac{1}{N} \sum_{j=1}^N \mathbb{1}_{\{Z_k^j = i\}} (1 - X_{k+1}^j)$$

and

$$p_N(i, k - i) = \frac{b_N(i, k - i)}{a_N(i, k - i)} .$$

This yields

$$\begin{aligned}
\hat{\theta} &= \arg \min_{\alpha, c} W_N(\alpha, c) \\
&= \arg \min_{\alpha, c} \sum_{k=0}^{n-1} \sum_{i=0}^k w_N(i, k-i) \{p_N(i, k-i) - f(\alpha, c, i, k-i)\}^2 \\
&= \arg \min_{\alpha, c} \sum_{k=1}^n \sum_{i=0}^{\lfloor k/2 \rfloor} w_N(i, k-i) \left\{ \frac{b_N(i, k-i)}{a_N(i, k-i)} - f(\alpha, c, i, k-i) \right\}^2 \\
&\quad + \sum_{k=1}^n \sum_{i=\lfloor k/2 \rfloor + 1}^k w_N(i, k-i) \left\{ \frac{a_N(i, k-i) - b_N(i, k-i)}{a_N(i, k-i)} - f(\alpha, c, k-i, i) \right\}^2 \\
&= \arg \min_{\alpha, c} \sum_{k=1}^n \sum_{i=0}^{\lfloor k/2 \rfloor} w_N(i, k-i) \left\{ \frac{b_N(i, k-i)}{a_N(i, k-i)} - f(\alpha, c, i, k-i) \right\}^2 \\
&\quad + w_N(k-i, i) \left\{ \frac{a_N(k-i, i) - b_N(k-i, i)}{a_N(k-i, i)} - f(\alpha, c, i, k-i) \right\}^2 \\
&= \arg \min_{\alpha, c} \sum_{k=1}^n \sum_{i=0}^{\lfloor k/2 \rfloor} \left\{ (w_N(i, k-i) + w_N(i-k, i)) f(\alpha, c, i, k-i)^2 \right. \\
&\quad \left. - 2(b_N(i, k-i) + a_N(k-i, i) - b_N(k-i, i)) f(\alpha, c, i, k-i) \right\} \\
&= \arg \min_{\alpha, c} \sum_{k=1}^n \sum_{i=0}^{\lfloor k/2 \rfloor} (w_N(i, k-i) + w_N(i-k, i)) \left\{ f(\alpha, c, i, k-i)^2 \right. \\
&\quad \left. - 2 \left(\frac{b_N(i, k-i) + a_N(k-i, i) - b_N(k-i, i)}{a_N(i, k-i) + a_N(k-i, i)} \right) f(\alpha, c, i, k-i) \right. \\
&\quad \left. + \left(\frac{b_N(i, k-i) + a_N(k-i, i) - b_N(k-i, i)}{a_N(i, k-i) + a_N(k-i, i)} \right)^2 \right\} \\
&= \arg \min_{\alpha, c} \sum_{k=1}^n \sum_{i=0}^{\lfloor k/2 \rfloor} (w_N(i, k-i) + w_N(i-k, i)) \left\{ f(\alpha, c, i, k-i) - \tilde{p}_N(i, k-i) \right\}^2.
\end{aligned}$$

5.4 Proofs for the RRW model

Proof of Theorem 3.2. Let us first check the identifiability condition (ii) of Assumption 2.1. By elementary computations, we have, for $0 \leq i \leq k \leq n-1$,

$$\begin{aligned}
f(\alpha, c, i, k-i) = f(\alpha_0, c_0, i, k-i) &\Leftrightarrow \left(\frac{c+i}{c+k-i} \right)^\alpha = \left(\frac{c_0+i}{c_0+k-i} \right)^{\alpha_0} \\
&\Leftrightarrow \frac{\alpha}{\alpha_0} = \frac{\log(c_0+i) - \log(c_0+k-i)}{\log(c+i) - \log(c+k-i)}. \tag{5.5}
\end{aligned}$$

Plugging the pairs $(i, k) = (0, 1)$ and $(i, k) = (0, 2)$ into (5.5) yields

$$\frac{\log(c_0) - \log(c_0 + 1)}{\log(c) - \log(c + 1)} = \frac{\log(c_0) - \log(c_0 + 2)}{\log(c) - \log(c + 2)} ,$$

or equivalently

$$\frac{\log(1 + 1/c_0)}{\log(1 + 2/c_0)} = \frac{\log(1 + 1/c)}{\log(1 + 2/c)} . \quad (5.6)$$

It is easily checked that the function $x \rightarrow \log(1 + x)/\log(1/2x)$ is strictly increasing on $(0, \infty)$. Thus (5.6) implies that $c = c_0$. Plugging this equality into (5.5) yields $\alpha = \alpha_0$. This proves Assumption 2.1-(ii).

We now prove that if $n \geq 2$, the vectors $\{\partial_\alpha f(\theta_0, i, k - i), 0 \leq i \leq k \leq n - 1\}$ and $\{\partial_c f(\theta_0, i, k - i), 0 \leq i \leq k \leq n - 1\}$ are linearly independent in $\mathbb{R}^{n(n-1)}$. For $0 \leq i \leq k \leq n - 1$, we have,

$$\begin{aligned} \partial_\alpha f(\alpha, c, i, k - i) &= f(\alpha, c, i, k - i) f(\alpha, c, k - i, i) \log \left(\frac{c + i}{c + k - i} \right) , \\ \partial_c f(\alpha, c, i, k - i) &= f(\alpha, c, i, k - i) f(\alpha, c, k - i, i) \frac{\alpha(k - 2i)}{(c + i)(c + k - i)} . \end{aligned}$$

Let $(u, v) \in \mathbb{R}^2$ and assume that for all $i, j \leq n - 1$ such that $i + j \leq n - 1$, it holds that

$$u \log \left(\frac{c_0 + i}{c_0 + j} \right) + v \frac{\alpha_0(j - i)}{(c_0 + i)(c_0 + j)} = 0 .$$

Replacing (i, j) for instance successively by $(0, 1)$ and $(0, 2)$ yields

$$\begin{cases} u \log \left(\frac{c_0}{c_0 + 1} \right) + v \frac{\alpha_0}{c_0(c_0 + 1)} = 0 , \\ u \log \left(\frac{c_0}{c_0 + 2} \right) + v \frac{2\alpha_0}{c_0(c_0 + 2)} = 0 . \end{cases}$$

If $(u, v) \neq (0, 0)$, this implies

$$\frac{c_0 + 2}{c_0} \log \left(\frac{c_0 + 2}{c_0} \right) + 2 \frac{c_0 + 1}{c_0} \log \left(\frac{c_0 + 1}{c_0} \right) = 0 .$$

By strict convexity of the function $x \rightarrow x \log x$ on $(0, \infty)$, this is impossible. Thus $u = v = 0$ and Assumption 2.1-(iii) holds. \square

Proof of Theorem 3.4, case $\alpha_0 = 1$. In this case the model is Polya's urn, and we have

$$\mathcal{I}_n(c) = \sum_{k=0}^{n-1} \frac{1}{2c + k} \left\{ \mathbb{E} \left[\frac{1}{c + Z_k} \right] + \mathbb{E} \left[\frac{1}{c + k - Z_k} \right] - \frac{4}{2c + k} \right\} . \quad (5.7)$$

The distribution of Z_k is given by

$$\mathbb{P}(Z_k = i) = \binom{k}{i} \frac{c(c+1) \cdots (c+i-1) \times c(c+1) \cdots (c+k-i-1)}{2c(2c+1) \cdots (2c+k-1)}.$$

Thus,

$$\mathbb{E} \left[\frac{1}{c + Z_k} \right] = \sum_{i=0}^k \binom{k}{i} \frac{c(c+1) \cdots (c+i-1) \times c(c+1) \cdots (c+k-i-1)}{2c(2c+1) \cdots (2c+k-1)} \frac{1}{c+i}.$$

For any $c > 0$, there exists constants $C_1 < C_2$ such that, for all integers $h \geq 1$,

$$C_1 h^c \leq \prod_{i=1}^h (1 + c/i) \leq C_2 h^c.$$

Therefore, there exists a constant $C > 0$ such that for all $k \geq 1$,

$$\mathbb{E} \left[\frac{1}{c + Z_k} \right] \leq C k^{-2} \sum_{i=1}^{k-1} \left(\frac{i}{k} \right)^{c-2} \left(1 - \frac{i}{k} \right)^{c-1} = \begin{cases} O(k^{-1}) & \text{if } c > 1, \\ O(k^{-1} \log k) & \text{if } c = 1, \\ O(k^{-c}) & \text{if } c < 1. \end{cases}$$

In all three cases, we obtain that the first series in (5.7) is summable. By symmetry, the sum of the second expectations is also finite. \square

Proof of Theorem 3.4, case $\alpha_0 < 1$. In this case, we know by Theorem 3.1 that Z_n/n converges almost surely to $1/2$. This implies that $f(\theta, Z_n, n - Z_n)$ converges almost surely to $1/2$ for all θ . By Cesaro's Lemma, this implies that $n^{-1} \ell_n(\theta) \rightarrow -\log 2$ a.s. \square

Proof of Theorem 3.4, case $\alpha_0 > 1$. Let Ω_1 be the event that color 1 is eventually selected, which happens with probability $1/2$ by Theorem 3.1. Then, on Ω_1 , $Z_n/n \rightarrow 1$ and if $k > T_\infty$, then $X_{k+1} = 1$ and $Z_k = k - Q_\infty$. Thus for large enough n , the log-likelihood on one path becomes

$$\begin{aligned} \ell_n(\theta) &= \sum_{k=0}^{T_\infty} X_{k+1} \log f(\theta, Z_k, k - Z_k) + (1 - X_{k+1}) \log \{1 - f(\theta, Z_k, k - Z_k)\} \\ &\quad + \sum_{k=T_\infty+1}^n \log f(\theta, k - Q_\infty, Q_\infty). \end{aligned}$$

As $k \rightarrow \infty$, for any $\alpha > 0$,

$$\log f(\theta, k - Q_\infty, Q_\infty) = -\log \left\{ 1 + \frac{(c + Q_\infty)^\alpha}{(c + k - Q_\infty)^\alpha} \right\} \sim -\frac{(c + Q_\infty)^\alpha}{(c + k - Q_\infty)^\alpha}.$$

If $\alpha \leq 1$ the series is divergent and thus $\lim_{n \rightarrow \infty} \ell_n(\theta) = -\infty$. If $\alpha > 1$ then the series is convergent and thus, on Ω_1 ,

$$\begin{aligned} \lim_{n \rightarrow \infty} \ell_n(\theta) &= \sum_{k=0}^{\infty} X_{k+1} \log f(\theta, Z_k, k - Z_k) + (1 - X_{k+1}) \log\{1 - f(\theta, Z_k, k - Z_k)\} \\ &= \sum_{k=0}^{T_\infty} X_{k+1} \log f(\theta, Z_k, k - Z_k) + (1 - X_{k+1}) \log\{1 - f(\theta, Z_k, k - Z_k)\} \\ &\quad + \sum_{T_\infty+1}^{\infty} \log f(\theta, k - Q_\infty, Q_\infty) . \end{aligned}$$

This implies that $\arg \max_{\theta \in \Theta} \ell_n(\theta) = \arg \max_{\theta \in \Theta, \alpha > 1} \ell_n(\theta)$ and that this argmax is a random variable which is a function of the whole path, and does not depend on the true value θ_0 . \square

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References

- [CDF⁺01] S. Camazine, J.-L. Deneubourg, N.R. Franks, J. Sneyd, G. Theraulaz, and E. Bonabeau. *Self-organization in biological systems*. Princeton studies in complexity. Princeton University Press, Princeton, N.J., 2001.
- [CL09] C. Cotar and V. Limic. Attracting time for strongly reinforced walks. *The Annals of Applied Probability*, 19:1972–2007, 2009.
- [DAGP90] J.-L. Deneubourg, S. Aron, S. Goss, and J.M. Pasteels. The self-organizing exploratory pattern of the argentine ant. *Journal of Insect Behavior*, 3:159–168, 1990.
- [DCD86] D. Dacunha-Castelle and M. Duflo. *Probability and statistics. Vol. II*. Springer-Verlag, New York, 1986.
- [DFM02] E. Drinea, A. Frieze, and M. Mitzenmacher. Balls and bins models with feedback. In *Proceedings of the Thirteenth Annual ACM-SIAM Symposium on Discrete Algorithms*, SODA '02, pages 308–315. Society for Industrial and Applied Mathematics, 2002.

- [DRDH00] H. Drees, S. Resnick, and L. De Haan. How to make a hill plot. *The Annals of Statistics*, 28:1–335, 2000.
- [GGC⁺09] S. Garnier, A. Guérécheau, M. Combe, V. Fourcassié, and G. Theraulaz. Path selection and foraging efficiency in argentine ant transport networks. *Behavioral Ecology and Sociobiology*, 63:1167–1179, 2009.
- [KK01] K. Khanin and R. Khanin. A probabilistic model for the establishment of neuron polarity. *Journal of Mathematical Biology*, 42:26–40, 2001.
- [Lim03] V. Limic. Attracting edge property for a class of reinforced random walks. *The Annals of Probability*, 31:1615–1654, 2003.
- [Pem07] R. Pemantle. A survey of random processes with reinforcement. *Probability Surveys*, 4:1–79, 2007.
- [PP12] S. Prawesh and B. Padmanabhan. Probabilistic news recommender systems with feedback. In *Proceedings of the Sixth ACM Conference on Recommender Systems*, RecSys ’12, pages 257–260. ACM, 2012.
- [Res07] S. Resnick. *Heavy-Tail Phenomena*. Springer series in operations research and financial engineering. Springer, 2007.
- [Tar11] P. Tarrès. Localization of reinforced random walks, 2011.
- [Was04] L. Wasserman. *All of Statistics: A Concise Course in Statistical Inference*. Springer, 2004.